DEPARTMENT OF COMPUTER SCIENCE
Cert
UNIVERSITY OF COPENHAGEN

Bridge Simulation on Lie Groups and Homogeneous Spaces

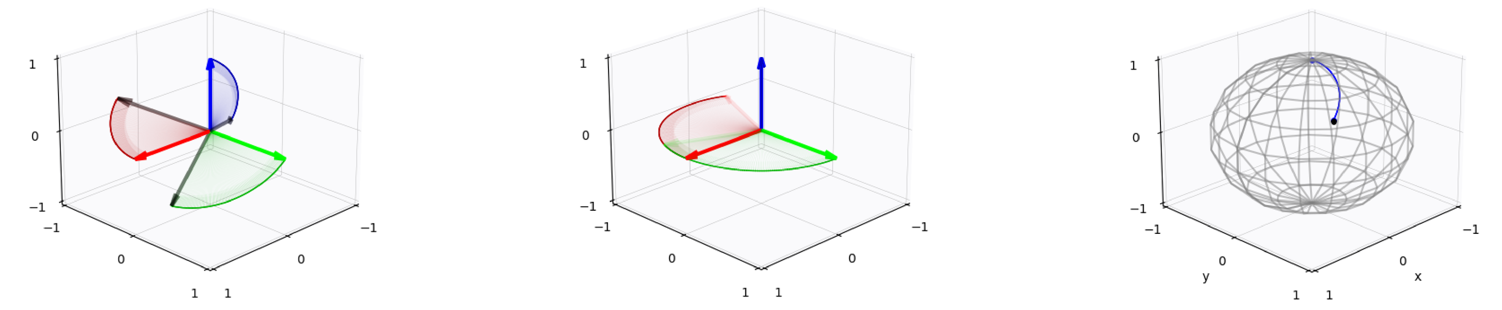
Mathias Højgaard Jensen
matje@di.ku.dk

Summary

- Diffusion bridge simulation is a non-trivial problem due to the intractability of the transition density of the underlying process. In the case of Lie group-valued data, assuming the data originating from some underlying unknown process, bridge simulation offers a method to estimate the underlying parameters, e.g., the *diffusion-mean* and the underlying metric structure used to generate the data. Certain manifolds can be described as homogeneous spaces, i.e., quotients of Lie groups. Conditioning on fibers in the Lie group leads to conditioning in the homogeneous space and essentially to parameter estimation in the quotient space.

Rotation Group

For illustration purposes, we restrict attention to the three-dimensional rotation group. The rotation group is a common example of a finite-dimensional matrix Lie group. There is a natural projection $\pi\colon SO(3) \to S^2$. This is visualized below.



(a) Action of the three-dimensional rotation group $SO(3)$ on three unit vectors (red,blue,green) in $\mathbb{R}^3$.

(b) The two-dimensional rotation group $SO(2)$ is a Lie subgroup of $SO(3)$.

(c) The two-dimensional sphere as a homogeneous space $SO(3)/SO(2)$, after

Figure 1: For each point p on the sphere S^2, the fiber $\pi^{-1}(p)$ is the rotation group $SO(2)$. Fixing the axis (0,0,1), the corresponding fiber is illustrated in 1b.

Simulation of Brownian Motion

A Brownian motion on a d-dimensional Lie group G is a solution to the stochastic differential equation (SDE)

$$dX_t = -\frac{1}{2}V_0(X_t)dt + V_i(X_t) \circ dB_t^i, \qquad X_0 = e, \tag{1}$$

where V_i's are left-invariant vector fields, and B^i are independent real-valued Brownian motions.

Numerical Simulations The Euler-Heun scheme approximates Stratonovich integration. Let $0 = t_0 < t_1 < \ldots < t_k = 1$, $\Delta t = t_k - t_{k-1}$, and $\Delta B_{t_i} \sim N(0, \Delta t)$, the numerical approximation of the Brownian motion (1) takes the form

$$x_{t_{k+1}} = x_{t_k} - \frac{1}{2}\sum_{j,i} C_{ij}^{j} V_i(x_{t_k})\Delta t + \frac{v_{t_{k+1}} + V_i(v_{t_{k+1}} + x_{t_k})\Delta B_{t_k}^i}{2} \tag{2}$$

where $v_{t_{k+1}} = V_i(x_{t_k})\Delta B_{t_k}^i$ is an intermediate value, and the C_{ij}^k are the structure coefficients. In the case of $SO(3)$, the structure coefficients are the Levi-Civita symbols.

(a) Brownian motion on $SO(3)$.

(b) Stochastic process on the quotient space S^2.

Figure 2: If the underlying metric is bi-invariant, then the resulting process on S^2 is also a Brownian motion.

Simulation of Brownian Bridges

Adding a pulling term to (1) yields a guided SDE: with the Lie group logarithm $\log\colon G \to T_eG$ and left-translation map $L_x\colon G \to G$, with derivative $d(L_x)\colon T_eG \to T_xG$, define $\log_y(v) = d(L_y)(\log(v))$

$$dY_t = -\frac{1}{2}V_0(X_t)dt + V_i(Y_t) \circ \left(dB_t^i - \frac{\log_{Y_t}(v)^i}{T-t}dt\right), \qquad Y_0 = e, \tag{3}$$

Numerical Bridge Sampling Algorithm on (3) Utilizing the simple expressions for the structure coefficients and the Lie group logarithmic map, we can explicitly write up the numerical approximation of the guided bridge processes (Brownian bridge) on (3) as

$$x_{t_{k+1}} = x_{t_k} - \frac{1}{2}\sum_{j,i}\epsilon^{ijj} V_i(x_{t_k})\Delta t + \frac{v_{t_{k+1}} + V_i(v_{t_{k+1}} + x_{t_k})\left(\Delta B_{t_k}^i - \frac{\log(x_k)}{T-t_k}\Delta t\right)}{2}, \tag{4}$$

where in this case we have $v_{t_{k+1}} = V_i(x_{t_k})\left(\Delta B_{t_k}^i - \frac{\log(x_k)}{T-t_k}\Delta t\right)$. A sample path of the guided bridge is visualized below.

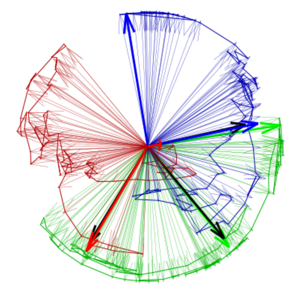
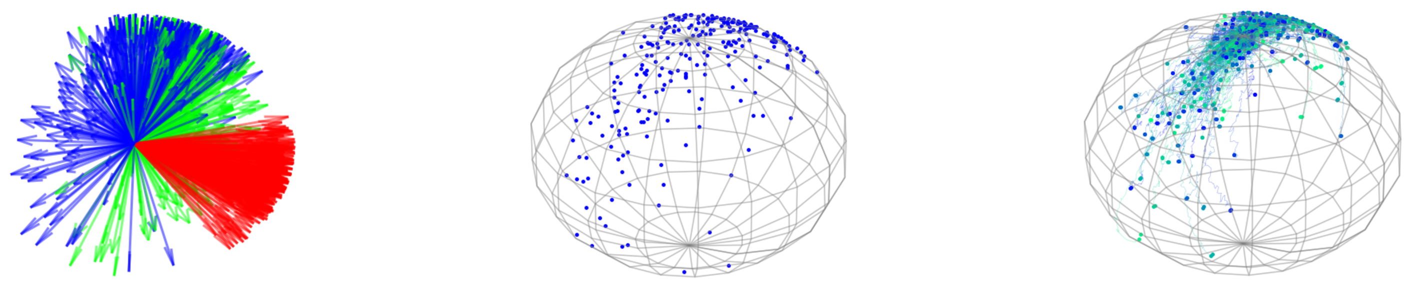


Figure 3: $\mathcal{M} = SO(3)$. The figure illustrates a sample path from the guided bridge on the rotation group. The black arrows indicate the conditioned point.

Application to Parameter Estimation

Metric Estimation on $SO(3)$ Suppose given data on $SO(3)$ is obtained as endpoints of Brownian motions generated from an unknown metric.



(a) Data on $SO(3)$

(b) The projected data on S^2

(c) Bridges on $SO(3)$ visualized through its projection onto S^2.

(d) Iterative MLE: Initial guess of the metric: Identity matrix.

(e) $\mathcal{M} = \mathbb{T}^2$. Four sample paths from the simulation scheme of the radial bridge, X_t, from x (red point) to v (black point).

Figure 4: Sampling k-bridges to each data point yields an approximation of the log-likelihood. An iterative maximum likelihood estimation (MLE) procedure provides an estimate of the underlying unknown metric.

Diffusion Mean Estimation on $S^2 = SO(3)/SO(2)$**.** Suppose given data on S^2 as in 5a. Sampling bridges in $SO(3)$ conditioned on the fiber over each data point yields a conditioning on S^2.

(a) Data sample on S^2.

(b) Estimated density in $SO(3)$ projected onto S^2.

(c) Convergence to the diffusion mean (blue point) on S^2 from initial guess (black point).

(d) Convergence of the likelihood.

Figure 5: Bridge simulation can be used for density estimation. To that effect it serves as a method for estimating the diffusion mean on S^2.

References

Diffusion mean in geometric spaces, Hansen, Eltzner, Huckemann, and Sommer, GSI 2021.
Bridge Simulation and Metric Estimation on Lie Groups, Jensen, Joshi, and Sommer, GSI 2021.
Bridge Simulation and Metric Estimation on Lie Groups and Homogeneous Spaces, Jensen, Hilgendorf, Joshi, and Sommer, (In preparation) 2021